Stress-gradient materials: an analytical exploration

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Workgroup on Computational Mechanics of Generalized Continua and Applications to Materials with Microstructure

Stress-gradient materials: an analytical exploration

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Motivation (1)

At first glance...

Strain gradient elasticity theory

$$\underbrace{w}_{\text{strain energy density}} = w(\underbrace{\varepsilon}_{\text{strain}}, \underbrace{\nabla\varepsilon}_{\text{strain gradient}})$$

Mindlin, R.D., Eshel, N.N., 4, pp. 109–124, International Journal of Solids and Structures, 1968

Motivation (2)

At first glance...

Strain gradient elasticity theory

$$w = w(\,\boldsymbol{\varepsilon}\,,\,\boldsymbol{\nabla}\boldsymbol{\varepsilon}\,)$$

Stress gradient elasticity theory

$$\underbrace{w^*}_{\text{stress energy density}} = w^*(\,\underbrace{\boldsymbol{\sigma}}_{\text{stress}},\,\underbrace{\boldsymbol{\nabla}\boldsymbol{\sigma}}_{\text{stress gradient}}\,)$$

S.Forest, K. Sab Mechanics Research Communications, 40, pp. 16–25, 2012

Motivation (3)

At first glance...

Strain gradient elasticity theory

$$w = w(\boldsymbol{\varepsilon}, \boldsymbol{\nabla}\boldsymbol{\varepsilon})$$

Stress gradient elasticity theory

$$w^* = w^*(\boldsymbol{\sigma}, \boldsymbol{\nabla}\boldsymbol{\sigma})$$

Is stress gradient elasticity equivalent or complementary to strain gradient elasticity?

S.Forest, K. Sab Mechanics Research Communications, 40, pp. 16–25, 2012

Objective

Outline

? How does stress gradient elasticity theory (Forest and Sab, 2012) differ from strain gradient one?

- Stress gradient elasticity theory (Forest and Sab, 2012).
- Closed form solution to Eshelby spherical inhomogeneity problem.
- Homogenization of heterogeneous stress-gradient materials (Mori Tanaka estimate)

S. Forest, K. Sab Mechanics Research Communications, 40, pp. 16–25, 2012
K. Sab, F. Legoll, S. Forest accepted for publication, Journal of Elasticity, 2015

Decomposition of stress gradient tensor

The spherical and deviatoric part of a third order tensor

Strain gradient tensor

- "free"

Stress gradient tensor

- constrained

$$\nabla \cdot \boldsymbol{\sigma} + \mathrm{b} = 0 \Leftrightarrow (\nabla \boldsymbol{\sigma}) : \boldsymbol{\delta} + \mathrm{b} = 0$$

Decomposition of stress gradient tensor

$$\nabla \boldsymbol{\sigma} = \mathbf{J} \therefore \nabla \boldsymbol{\sigma} + \mathbf{K} \therefore \nabla \boldsymbol{\sigma}$$

spherical projector ($\mathbf{J}$)   deviatoric projector ($\mathbf{K}$)

$$J_{ijklmn} = \frac{1}{8} \left(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} \right)$$

$$K_{ijklmn} = I_{ijklmn} - J_{ijklmn}$$

Stress energy density function

$$w^* = w^*(\boldsymbol{\sigma}, \underbrace{\mathbf{K} \therefore \boldsymbol{\nabla\sigma}}_{\mathbf{R}})$$

For isotropic stress gradient materials:

$$w^*(\boldsymbol{\sigma}, \mathbf{R}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{S} : \boldsymbol{\sigma} + \frac{1}{2}\mathbf{R} \therefore \mathbf{M} \therefore \mathbf{R}$$

Constitutive laws

$$\mathbf{e} = \frac{\partial w^*}{\partial \boldsymbol{\sigma}} = \mathbf{S} : \boldsymbol{\sigma}, \quad \boldsymbol{\Phi} = \frac{\partial w^*}{\partial \mathbf{R}} = \mathbf{M} \therefore \mathbf{R}$$

NOTA: $\boldsymbol{\Phi}, \mathbf{R}$ have to be deviatoric tensors!

$$\mathbf{M} = \mathbf{K} \therefore \mathbf{M} \therefore \mathbf{K}$$

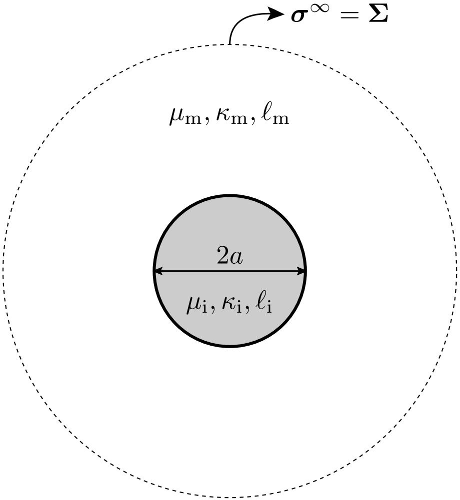
A comparison of field equations (no body force)

	Simplified stress gradient	Simplified strain gradient
Equilibrium	$\sigma_{ij,j} = 0$	$\tau_{ij,j} = 0$
Gradient	$R_{ijk} = \sigma_{ij,k}$	$\kappa_{ijk} = \varepsilon_{ij,k}$
Constitutive laws	$e_{ij} = S_{ijkl}\sigma_{kl}$ $e_{ij} = \varepsilon_{ij} + \Phi_{ijk,k}$ $\Phi_{ijk} = \ell^2 K_{ijkpqr} S_{pqmn} R_{mnr}$	$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ $\tau_{ij} = \sigma_{ij} - \mu_{ijk,k}$ $\mu_{ijk} = \ell^2 C_{ijmn}\kappa_{mnk}$
SUBC	$\sigma_{ij} = \Sigma_{ij}$	$\sigma_{ij}n_j - (\mu_{ijk}n_k)_{,j} + (\mu_{ijk}n_k n_l)_{,l} n_j = t_i$ on $\partial\Omega$ (smooth part) $\mu_{ijk}n_j n_k = q_i$ on $\partial\Omega$ (smooth part) $[[\mu_{ijk}m_j n_k]] = 0$ on the edge of $\partial\Omega$

S.Forest, K. Sab Mechanics Research Communications, 40, pp. 16–25, 2012
Altan, B.S., Aifantis, E.C., Journal of the Mechanical Behavior of Materials, 8(3), pp. 231–282, 1997
Gao, X.-L., Park, S.K. International Journal of Solids and Structures, 44, pp. 7486–7499, 2007

Eshelby's spherical inhomogeneity problem (1)

Geometry + mechanical properties



Stress gradient (this work):

- Uniform isotropic stress:

$$\boldsymbol{\Sigma} = \mathbf{e}_x \otimes \mathbf{e}_x + \mathbf{e}_y \otimes \mathbf{e}_y + \mathbf{e}_z \otimes \mathbf{e}_z$$

- Uniform axial stress:

$$\boldsymbol{\Sigma} = \mathbf{e}_z \otimes \mathbf{e}_z$$

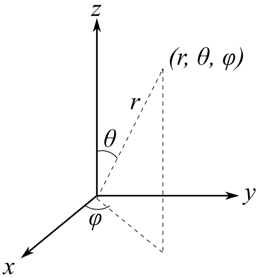
Strain gradient:

- **Gao, X.-L., Ma, H.M** Acta Mechanica, 207 (3), pp. 163-181, 2009
- **Gao, X.-L., Ma, H.M** Journal of the Mechanics and Physics of Solids, 58 (5), pp. 779-797, 2010

Eshelby's spherical inhomogeneity problem (2)

Closed-form solution

Postulated stress form in spherical coordinates (similar to Love's approach)



- Uniform isotropic stress prescribed at boundary:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(r)$$

- Uniform axial stress prescribed at boundary:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(r, \theta)$$

Continuity conditions at the surface matrix-inclusion

$$\boldsymbol{\sigma}_{\mathsf{m}}(r = a) = \boldsymbol{\sigma}_{\mathsf{i}}(r = a)$$

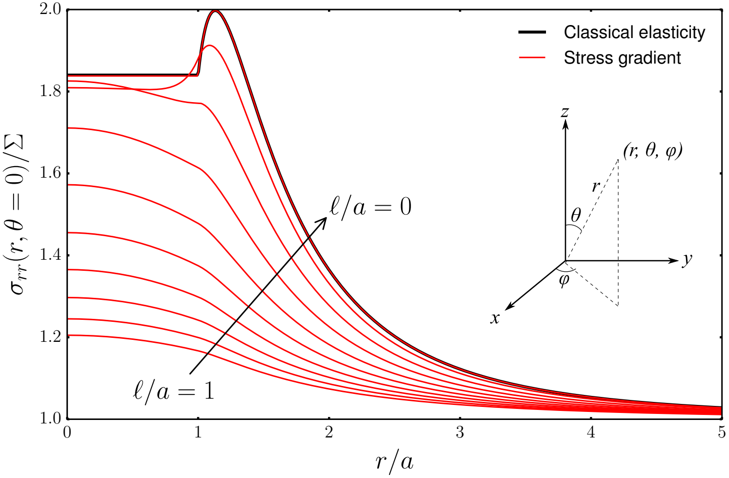
$$\left[\mathbf{u} \otimes^{\mathsf{s}} \mathbf{n} + \boldsymbol{\Phi} \cdot \mathbf{n}\right]_{\mathsf{m}} (r = a) = \left[\mathbf{u} \otimes^{\mathsf{s}} \mathbf{n} + \boldsymbol{\Phi} \cdot \mathbf{n}\right]_{\mathsf{i}} (r = a)$$

Love, A.E.H. Dover Publications, New York, 1944

Uniaxial problem $\mathbf{\Sigma} = \mathbf{e}_z \otimes \mathbf{e}_z$

Radial stress field $\sigma_{rr}(r, \theta = 0)$

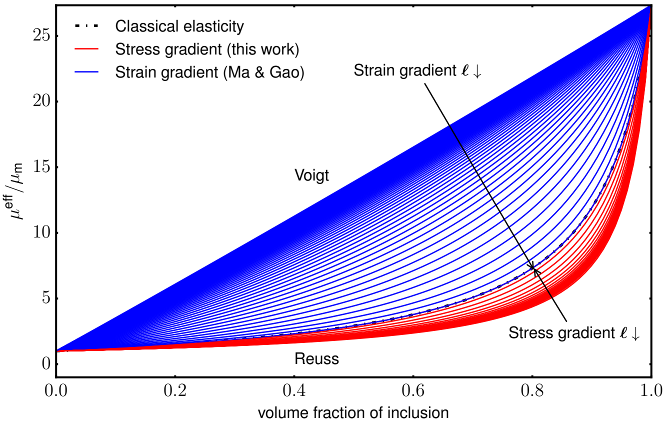
$$\mu_\mathrm{i} = 27\mu_\mathrm{m}, \nu_\mathrm{i} = 0.4, \nu_\mathrm{m} = 0.08, \ell_\mathrm{i} = \ell_\mathrm{m} = \ell$$



Effect of material internal length (1)

Effective shear moduli (Mori Tanaka estimate)

$$\mu_{\mathrm{i}} = 27\mu_{\mathrm{m}}, \nu_{\mathrm{i}} = 0.4, \nu_{\mathrm{m}} = 0.08, \ell_{\mathrm{i}} = \ell_{\mathrm{m}} = \ell$$



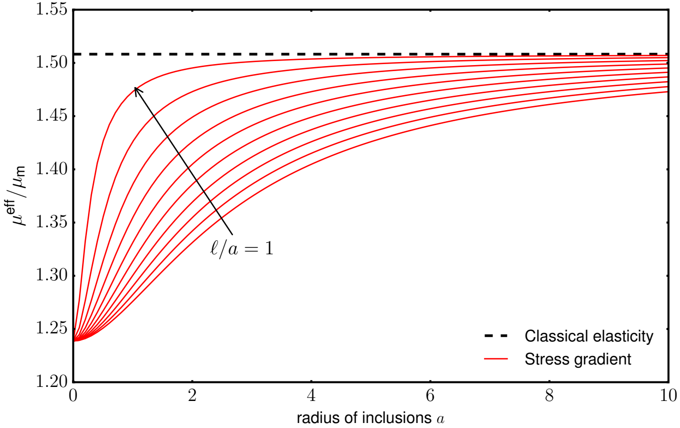
Stress gradient is complementary to strain gradient elasticity!

Ma, H.M, Gao, X.-L. Acta Mechanica, 225 (4-5), pp. 1075-1091, 2014

Effect of material internal length (2)

Effective shear moduli: $f = 20\%$

$$\mu_\mathrm{i} = 27\mu_\mathrm{m}, \nu_\mathrm{i} = 0.4, \nu_\mathrm{m} = 0.08, \ell_\mathrm{i} = \ell_\mathrm{m} = \ell$$



- **Stress gradient elasticity: capture size effects.**
- **For $a \gg \ell$, stress gradient effects vanish as expected.**

Softening size effect

... has been reported elsewhere (molecular dynamics simulations)!

To cite some:

- Polyimide matrix *"reinforced"* by silica spherical nanoparticles with various surface treatment.

  Odegard, G.M., Clancy, T.C., Gates, T.S. Polymer, 46, pp.553–562, 2005

- Polystyrene matrix *"reinforced"* by silica spherical nanoparticles.

  Davydov, D., et al Soft Materials, 12, S142–S151, 2014

- etc

Conclusion & Perspective

Conclusion

- Solution to Eshelby's spherical inhomogeneity problem
- Mori Tanaka estimate for stress gradient materials
- Stress gradient elasticity differs from strain gradient elasticity
  - Strain gradient: stiffening size effect
  - Stress gradient: softening size effect

Perspective

- Numerical implementation: FEM, FFT
- Physical interpretation of material internal length?

Acknowledgments

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Thank you for your attention!

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